WS 2016/17

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Symplectic Geometry

Homework 8

Exercise 1. (10 points)
Prove the Arnold Conjecture for the $\mathcal{C}^1$ small case, i.e. prove that if M is a compact symplectic manifold and $\phi\colon M \to M$ is a Hamiltonian diffeomorphism, sufficiently close to the identity in $\mathcal{C}^1$ topology, then

$$\text{the number of fixed points of } \phi \geq \mathrm{Crit}(M)$$

where $\mathrm{Crit}(M)$ is the minimal number of critical points a smooth function $f\colon M \to \mathbb{R}$ may have.

Exercise 2. (10 points)
Exercise 2 from Homework 6 (page 50) in *Lectures on Symplectic Geometry* by A. Cannas da Silva.

Exercise 3. (10 points)
Recall the setting of the proposition where we prove that any symplectic vector space (V, Ω) has a compatible complex structure J. Let $A\colon V \to V$ be the linear isomorphism satisfying $\Omega(v, u) = G(Av, u)$. Prove:

(i) $J^T = -J$,

(ii) J is compatible with Ω.

Hint for (i): Show that A is skew symmetric and that it commutes with $(AA^T)^{-1/2}$ (so also with J). You may want to use the splitting of V into eigenspaces of AA^T.

Exercise 4. (10 points)
Let (V, Ω) be a symplectic vector space and J be a complex structure on V. Prove that the following are equivalent:

1. J is compatible with Ω.
2. The bilinear form $g_J\colon V \times V \to \mathbb{R}$ defined by
$$g_J(v, w) = \Omega(v, Jw)$$
is symmetric, positive definite and J-invariant.
3. The form $H\colon V \times V \to \mathbb{C}$ defined by
$$H(v, w) = \Omega(v, Jw) + i\Omega(v, w)$$
is complex linear in w, complex anti-linear in v, satisfies $H(v, w) = \overline{H(w, v)}$, and has a positive definite real part. Such a form is called a **Hermitian inner product** on (V, J). Here V is viewed as a vector space over $\mathbb{C}$, with the multiplication by i given by the action of J. Thus the condition that H is complex linear in w is understood as $H(v, Jw) = iH(v, w)$, etc.

Hand in: Thursday December 15th
in the exercise session
in Übungsraum 1, MI